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香 港 中 文 大 學

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二零二一至二二年度上學期科目考試

Course Examination 1st Term, 2021-22

科目編號及名稱
Course Code & Title : **MATH2020A Advanced Calculus II**

時間
Time allowed : **2** 小時 hours **00** 分鐘 minutes

學號
Student I.D. No : ______ 座號 Seat No. : ______

Answer all questions. You should justify your answer and show all details.

1. (10 points) Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$.

2. (15 points) Convert the iterated integral

$$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{\sqrt{2}}^{\sqrt{4-r^2}} zr\,dzdrd\theta\ ,$$

into (a) an iterated integral in $dzdxdy$ and (b) in spherical coordinates. Then evaluate this integral in any way you like.

3. (10 points) Evaluate

$$\iint_D \sqrt{x}y^2\,dA$$

where D is the region bounded by $xy = 1, xy = 8, y = \sqrt{x}, y = 5\sqrt{x},\ x, y \geq 0$.

4. (10 points) Let P be the parallelogram formed by the lines $x + y = -1, x + y = 2, y = 2x, y = 2x + 5$ and γ its boundary oriented in anticlockwise direction. Find the flux of the vector field $\mathbf{F} = xy\mathbf{i} + x\mathbf{j}$ across γ and the circulation of $\mathbf{F}$ around γ.

5. (10 points) Consider the cycloid $x = t - \sin t, y = 1 - \cos t\ , t \in [0, 2\pi]$. Find (a) its length and (b) the area enclosed by it and the x-axis.

6. (15 points) Let
$$\mathbf{H} = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j},$$
which is defined in the plane except at the origin.

   (a) Prove that $\mathbf{H}$ is not conservative in $\{(x,y):\ (x,y) \neq (0,0)\}$.

   (b) Prove that $\mathbf{H}$ is conservative in the right half plane $\{(x,y):\ x > 0\}$.

   (c) Find the work done by $\mathbf{H}$ along C, a smooth curve running from $(10,10)$ to $(\sqrt{3},1)$ in the right half plane.

7. (10 points) Let $\mathbf{G}$ be a smooth vector field defined in the plane away from the origin. Assume that it satisfies (a) $\mathbf{G}(\mathbf{r})$ points toward the origin at every point $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and (b) its magnitude only depends on its distance to the origin, that is, $|\mathbf{G}(\mathbf{r}_1)| = |\mathbf{G}(\mathbf{r}_2)|$ whenever $|\mathbf{r}_1| = |\mathbf{r}_2|$.

   (a) Show that $\mathbf{G}$ is of the form $g(|\mathbf{r}|)\mathbf{r}$ for some negative, smooth function g defined in $(0,\infty)$.

   (b) Show that $\mathbf{G}$ is conservative and find its potential function.

8. (10 points) Consider a simple curve sitting in the right half xz-plane parametrized by $x(t)\mathbf{i} + z(t)\mathbf{k},\ t \in [a,b]$. Rotate it around the z-axis to get a surface S.

   (a) Show that the surface area for S is given by
$$2\pi \int_a^b x(t)\sqrt{x'^2(t) + z'^2(t)}\, dt \ .$$

   (b) Find the surface area of the torus which is obtained by rotating the circle
$$(x-a)^2 + z^2 = r^2, \quad 0 < r < a \ ,$$
   around the z-axis.

9. (10 points) The plane $3x + 2y + z = 6$ intersects the coordinate axes to form a triangle in the first octant. Its boundary C is oriented in the anticlockwise way as viewed from above. Use Stokes' theorem to evaluate the line integral
$$\oint_C \mathbf{F} \cdot d\mathbf{r} \ ,$$
where $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} + xz\mathbf{k}$.

--- End of Paper ---

— Selected Solution —

$\boxed{4}$ By Green's th., $M = xy$, $N = x$

$$\text{flux} = \oint_\gamma N\,dx - M\,dy = \iint_P (M_x + N_y)\,dA = \iint_P y\,dA.$$

$$\text{Circulation} = \oint_\gamma M\,dx + N\,dy = \iint_P (N_x - M_y)\,dA = \iint_P (1-x)\,dA.$$

Now, let $u = x+y \in [-1, 2]$

$v = y - 2x \in [0, 5]$.

then $x = (u-v)/3$, $y = (2u+v)/3$.

$$\frac{\partial(u,v)}{\partial(x,y)} = 3.$$

$$\therefore \text{flux} = \int_0^5 \int_{-1}^2 \frac{1}{3}(2u+v)\,\frac{1}{3}\,du\,dv \cdots = \frac{35}{6}.$$

$$\text{Circulat.} = \int_0^5 \int_{-1}^2 \left(1 - \frac{1}{3}u + \frac{1}{3}v\right)\frac{1}{3}\,du\,dv = \cdots = \frac{25}{3}.$$

$\boxed{2}$ (a) $$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} \int_{\sqrt{2}}^{\sqrt{4-x^2-y^2}} z\,dz\,dx\,dy$$

(b) $$\int_0^{2\pi} \int_0^{\pi/4} \int_{\sqrt{2}/\cos\varphi}^{2} \rho^3 \cos\varphi \sin\varphi\,d\rho\,d\varphi\,d\theta.$$

(c) $$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_{\sqrt{2}}^{\sqrt{4-r^2}} z\,r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^{\sqrt{2}} \frac{z^2}{2}\Big|_{\sqrt{2}}^{\sqrt{4-r^2}} r\,dr\,d\theta$$

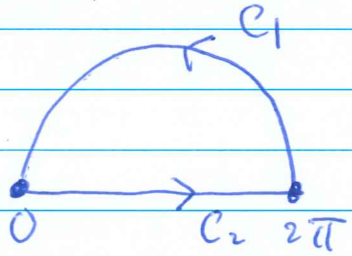
$$= \cdots = \pi \ \#$$

5 cycloid $x = t - \sin t$, $y = 1 - \cos t$, $t \in [0, 2\pi]$

(a) $x' = 1 - \cos t$, $y' = \sin t$

$$|\vec{r}'(t)| = \sqrt{x'(t)^2 + y'^2(t)}$$
$$= \sqrt{(1-\cos t)^2 + \sin^2 t}$$
$$= \sqrt{2(1-\cos t)}$$

$$\therefore \text{length} = \int_0^{2\pi} \sqrt{2(1-\cos t)}\, dt$$
$$= \sqrt{2}\int_0^{2\pi} \sqrt{2\sin^2 \tfrac{t}{2}}\, dt$$
$$= 2\int_0^{2\pi} \sin \tfrac{t}{2}\, dt = 4\left(-\cos \tfrac{t}{2}\right)\Big|_0^{2\pi}$$
$$= 8 \quad \#$$



(b) area.

$$= \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

$C = C_1 + C_2$

$$= \frac{1}{2}\int_{C_1} x\,dy - y\,dx + \frac{1}{2}\int_{C_2} x\,dy - y\,dx$$

$t\sin t - 1 - 1 \; + 2\cos t$

$$\int_{-C_1} x\,dy - y\,dx = \int_0^{2\pi} [(t-\sin t)(\sin t) - (1-\cos t)(1-\cos t)]\,dt$$
$$= \int_0^{2\pi} (t\sin t + 2\cos t - 2)\,dt$$
$$= t(-\cos t)\Big|_0^{2\pi} + \int_0^{2\pi} \cos t\,dt - 2\times 2\pi$$
$$= -2\pi - 4\pi = -6\pi$$

$= 3\pi$

$\oint$ $C: t \mapsto (t, 0)$, $t \in [0, 2\pi]$, $x'(t) = 1$, $y'(t) = 0$

$$\int_{C_2} x\,dy - y\,dx = \int_0^{2\pi} [x \times 0 - 0 \times]\,dt = 0$$

$\boxed{6}$ (a) Let $C : \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $t \in [0, 2\pi]$.

$$\oint_C \vec{H}\cdot d\vec{r} = \int_0^{2\pi} \cdots\cdots dt$$

$$= \int_0^{2\pi} dt$$

$$= 2\pi \neq 0.$$

As $\vec{H}$ is conservative iff $\oint_\gamma \vec{H}\cdot d\vec{r}$ for every simple, closed curve γ in $\mathbb{R}^2 \setminus \{(0,0)\}$. Now, $\oint_C \vec{H}\cdot d\vec{r} \neq 0$ for this C, so $\vec{H}$ is not conservative in $\mathbb{R}^2\setminus\{(0,0)\}$.

(b) Let $\mathbb{R}^2_+$ be the right half plane.

Let γ be a simple closed curve in $\mathbb{R}^2+$. It encloses D.

$$\oint_\gamma \vec{H}\cdot d\vec{r} = \iint_D (N_x - M_y)\,dA \qquad \text{Green's th.}$$

$$N_x = \cdots = \frac{y^2 - x^2}{(x^2+y^2)^2}, \quad M_y = \cdots \frac{y^2-x^2}{(\quad)^2}, \quad \therefore N_x = M_y$$

$$\therefore \oint_\gamma \vec{H}\cdot d\vec{r} = 0$$

$\therefore$ $\vec{H}$ is conservative in $\mathbb{R}^2+$.

Or, one may simply find the potential of $\vec{H}$.

$$\frac{\partial g}{\partial x} = \frac{-y}{x^2+y^2} \Rightarrow g(x,y) = \tan^{-1}\frac{y}{x} + h(y)$$

$$\frac{\partial g}{\partial y} = \frac{x}{x^2+y^2} + \frac{\partial h}{\partial y} \Rightarrow \frac{\partial h}{\partial y} = c.$$

$\therefore$ $\vec{H}$ admits a potential $\Phi(x,y) = \tan^{-1}\dfrac{y}{x}$

$\boxed{7}$ (a) Let $r = \sqrt{x^2+y^2+z^2} = |\vec{r}| \geq 0$.

Recall that for any vector $\vec{u}$, $\vec{u} = |\vec{u}| \frac{\vec{u}}{|\vec{u}|}$ where $\frac{\vec{u}}{|\vec{u}|}$ is a unit vector.

$$\therefore \quad G(\vec{r}) = |G(\vec{r})| \, \hat{\xi}(\vec{r}) \quad \text{where } |\hat{\xi}| = 1.$$

By assumption (a) $\hat{\xi}(\vec{r}) = -\vec{r}/r$.

By assumption (b) $|G(\vec{r})| = h(r)$ for some $h \geq 0$ in $(0, \infty)$.

$$\therefore \quad G(\vec{r}) = -h(r)\frac{\vec{r}}{r}, \text{ i.e.,}$$

$$g(r) = -h(r)/r.$$

(b) We find the potential for G.
Assume it is of the form $\Phi(\vec{r}) = \varphi(r)$.

$$\nabla\Phi(\vec{r}) = \varphi'(r)\frac{\vec{r}}{r}$$

We want $\nabla\Phi(\vec{r}) = G(\vec{r})$, ie

$$\varphi'(r)\frac{\vec{r}}{r} = -h(r)\frac{\vec{r}}{r}, \text{ ie}$$

$$\varphi'(r) = -h(r)$$

$$\therefore \quad \varphi(r) = \int_1^r -h(t)\,dt.$$

The potential for G is

$$\Phi(\vec{r}) = \int_1^r g(t)\, t \, dt \quad \#$$

$\boxed{8}$ ⓐ $(t, \varphi) \mapsto (x(t)\cos\varphi,\ x(t)\sin\varphi,\ z(t))$

parametrize the surface S.

$$\vec{r}_t = (x'\cos\varphi,\ x'\sin\varphi,\ z')$$

$$\vec{r}_\varphi = (-x\sin\varphi,\ x\cos\varphi,\ 0)$$

$$\vec{r}_t \times \vec{r}_\varphi = xz'\cos\varphi\,\hat{i} - xz'\sin\varphi\,\hat{j} + x'x\,\hat{k}$$

$$|\vec{r}_t \times \vec{r}_\varphi| = x\sqrt{x'^2+z'^2}$$

$\therefore$ surface area of S

$$= \iint_{[a,b]\times[0,2\pi]} x\sqrt{x'^2+z'^2}\,dt\,d\varphi = 2\pi\int_a^b x\sqrt{x'^2+z'^2}\,dt \quad \#$$

ⓑ The circle is parametrized by

$$t \mapsto (a + r\cos t,\ r\sin t),\quad t \in [0, 2\pi]$$

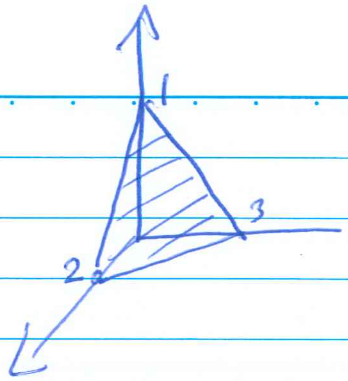
$$x' = -r\sin t,\quad z' = r\cos t,$$

$$\sqrt{x'^2+z'^2} = r$$

$\therefore$ surface area of torus $=$

$$2\pi\int_0^{2\pi} (a + r\cos t)\, r\,dt = 4\pi^2 ar. \quad \#$$

$\boxed{9}$



$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ xy & y & xz \end{vmatrix} = -z\hat{j} - x\hat{k}$$

The triangle T is described by

$(x, y, z = 6-3x-2y)$

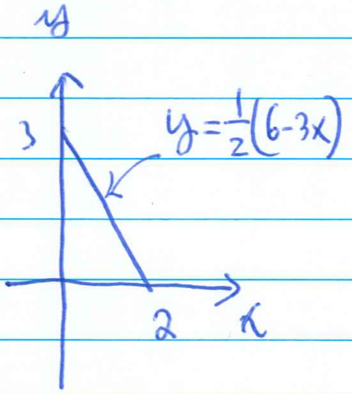
$r_x = (1,0,-3), \quad r_y = (0,1,-2)$

$r_x \times r_y = 3\hat{i} + 2\hat{j} + \hat{k}$

Stokes' thm

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_T \nabla\times\vec{F}\cdot\vec{r}_x\times\vec{r}_y \, dA$$

$$= \iint_T (-2z - x)\, dA(x,y)$$



$$= \iint_T (-12 + 6x + 4y - x)\, dA(x,y)$$

$$= \iint_T (-12 + 5x + 4y)\, dA(x,y)$$

$$= \int_0^2 \int_0^{\frac{1}{2}(6-3x)} (-12 + 5x + 4y)\, dy\, dx$$

$= -14$ #